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Arithmetic Properties of Certain Functions in Several Variables

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If T is an $n \times n$ matrix with nonnegative integral entries, we define a transformation $T: \mathbf{C}^n \to \mathbf{C}^n$ by $w = Tz$ where

$$w_i = \prod_{j=1}^{n} z_j^{t_{ij}} \qquad (1 \leqslant i \leqslant n).$$

We consider functions $f(z)$ of n complex variables which satisfy a functional equation giving $f(Tz)$ as a rational function of $f(z)$ and we obtain conditions under which such a function $f(z)$ takes transcendental values at algebraic points.

INTRODUCTION

In a sequence of three papers, "Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen," "Über das Verschwinden von Potenzreihen mehrerer Veränderlichen in speziellen Punktfolgen," and "Arithmetische Eigenschaften einer Klasse transzendental-transzendenter Funktionen," published more than 40 years ago, Mahler discussed arithmetic properties of functions in several complex variables satisfying a certain type of functional equation. Recently, in [9], Mahler gave a summary of his earlier work and raised a number of problems connected with it. The present investigation is concerned with one of these problems. Our aim is to extend the methods of the first two of Mahler's papers, [6, 7], mentioned above, so that the results apply to a somewhat larger class of functions.

The ingredients of the main theorem may be described as follows.

Let $T = (t_{ij})$ be an $n \times n$ matrix with nonnegative integer entries. If $z = (z_1, \ldots, z_n)$ is a point of $\mathbf{C}^n$, we define a transformation $T: \mathbf{C}^n \to \mathbf{C}^n$ by $w = Tz$ where $w = (w_1, \ldots, w_n)$ has coordinates

$$w_i = \prod_{j=1}^{n} z_j^{t_{ij}} \qquad (1 \leqslant i \leqslant n). \tag{1}$$

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We consider functions $F(z) = F(z_1, ..., z_n)$ of n complex variables which satisfy a functional equation of the shape

$$F(Tz) = \sum_{j=0}^{s} a_j(z)\, F(z)^j \Big/ \sum_{j=0}^{s} b_j(z)\, F(z)^j, \tag{2}$$

where the coefficients $a_j(z)$ and $b_j(z)$ are polynomials in z with algebraic coefficients. We shall prove that such a function takes a transcendental value at any point $\alpha = (\alpha_1, ..., \alpha_n)$ with algebraic coordinates, provided that the matrix T, the function F, and the point α satisfy a number of rather natural conditions which, however, it is not appropriate to detail at the moment.

The following examples are instances of the general result. The function

$$F(z) = \prod_{h=0}^{\infty} (1 - z^{2^h}) \qquad (z \text{ in } \mathbf{C})$$

satisfies the functional equation

$$F(z^2) = F(z)/(1 - z).$$

If α is a nonzero algebraic number with $|\alpha| < 1$, then $F(\alpha)$ is transcendental [6].

The function

$$F_k(z) = \sum_{h=0}^{\infty} z^{k^h}/(1 - z^{k^h}) \qquad (z \text{ in } \mathbf{C}),$$

where $k \geqslant 2$ is an integer, satisfies the functional equation

$$F_k(z^k) = F_k(z) - z/(1 - z).$$

If α is a nonzero algebraic number with $|\alpha| < 1$, then $F_k(\alpha)$ is transcendental [9].

Let ω be a positive quadratic irrational and suppose the simple continued fraction of ω is purely periodic. Denote the period of this continued fraction by h and its convergents by p_k/q_k $(k = 0, 1, 2, ...)$. Let T be the matrix

$$T = \begin{pmatrix} p_h & q_h \\ p_{h-1} & q_{h-1} \end{pmatrix}.$$

The function

$$F_\omega(z) = F_\omega(z_1, z_2) = \sum_{k=1}^{\infty} \sum_{l=1}^{[k\omega]} z_1^k z_2^l \qquad (z_1, z_2 \text{ in } \mathbf{C})$$

satisfies a functional equation of the shape

$$F_\omega(Tz) = (-1)^h F_\omega(z) + R(z),$$

where $R(z)$ is a certain rational function in z_1 and z_2 with integer coefficients. If α_1, α_2 are nonzero algebraic numbers such that

$$\log|\alpha_1| + \omega \log|\alpha_2| < 0 \qquad \text{and} \qquad \alpha_1^{p_k}\alpha_2^{q_k} \neq 1 \qquad (k = 0, 1, 2,...),$$

then $F_\omega(\alpha_1, \alpha_2)$ is transcendental. In particular,

$$F_\omega(\alpha, 1) = \sum_{k=1}^{\infty} [k\omega]\, \alpha^k$$

is a transcendental number whenever α is algebraic and $0 < |\alpha| < 1$ [6].

The paper is divided into three parts. Part I contains a number of definitions and preliminary lemmas concerning nonnegative matrices, the transformation (1), and the functional equation (2), which will be needed in the statements and proofs of the theorems. The transcendence theorem itself is stated and proved in Part II. In the notation introduced above, the hypotheses of the theorem may be roughly described as regularity conditions imposed on the matrix T, the function F, and the algebraic point α under consideration, together with the requirement that the point α have the following property which we call Property (A):

(A) If $f(z)$ is any function of n complex variables which is regular in some neighborhood of the origin and not identically zero, then there are infinitely many natural numbers k such that $f(T^k\alpha) \neq 0$.

We examine this condition separately in Part III and obtain some simple criteria, though probably not the best possible ones, which guarantee that Property (A) holds.

We would like to express our thanks to Mahler for drawing our attention to his papers [6–9] and to acknowledge the very considerable debt we owe to the ideas in them.

I. Definitions and Preliminary Lemmas

1. *A Class of Nonnegative Matrices*

Let X be a matrix (or vector). If all entries of X are nonnegative, we call X a nonnegative matrix (or vector) and write $X \geqslant 0$, and if all the entries of X are positive, we call X a positive matrix (or vector) and write $X > 0$. In the same way, we shall often speak of integral matrices, algebraic points and so on.

Let $T = (t_{ij})$ be an $n \times n$ nonnegative matrix. We call T reducible if the indices $1, 2,..., n$ can be divided into two disjoint nonempty sets $i_1,..., i_\mu$ and $j_1,..., j_\nu$ $(\mu + \nu = n)$ such that

$$t_{i_p j_q} = 0 \qquad (1 \leqslant p \leqslant \mu,\ 1 \leqslant q \leqslant \nu).$$

Otherwise, T is irreducible. Thus T is reducible if and only if there is a permutation of $\{1, 2,..., n\}$ which, when applied to the rows and columns of T, reduces it to the form

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix},$$

where A and C are square matrices.

A nonnegative square matrix T has a normal form (obtained by a permutation of the indices) of the shape

$$T = \begin{bmatrix} T_1 & & & & & & \\ & T_2 & & 0 & & & \\ & & \ddots & & & & \\ & 0 & & & & & \\ & & & T_\kappa & & & \\ T_{\kappa+1,1} & T_{\kappa+1,2} & \cdots & & T_{\kappa+1} & & \\ \vdots & & & & & \ddots & \\ T_{\nu,1} & T_{\nu,2} & & \cdots & & & T_\nu \end{bmatrix} \tag{1}$$

where $T_1 ,..., T_\nu$ are irreducible square matrices and, for each α ($\kappa + 1 \leqslant \alpha \leqslant \nu$), not all the $T_{\alpha,\beta}$ ($1 \leqslant \beta \leqslant \alpha - 1$) are 0. (See [3, pp. 89–92]). The normal form is unique up to certain permutations of the blocks and certain permutations of the indices within the blocks.

As usual, we define the spectral radius of a square matrix T to be the maximum of the absolute values of the eigenvalues of T; we denote the spectral radius of T by $r(T)$. It will be necessary for us to restrict the transformation matrices which we consider to the class $\mathscr{T}$, defined as follows.

Definition 1. An $n \times n$ matrix T is of class $\mathscr{T}$ if its entries are nonnegative integers and its normal form (1) has the following two properties:

(i) $r(T_1) = r(T_2) = \cdots = r(T_\kappa) = r(T)$, and

(ii) $r(T_\rho) < r(T)$ $(\kappa + 1 \leqslant \rho \leqslant \nu)$.

The first two lemmas are known results on nonnegative matrices which we shall need in the discussion.

Lemma 1 (Frobenius). *Let T be an irreducible nonnegative matrix with spectral radius r. Then r is a simple eigenvalue of T. Further, if T has precisely h eigenvalues $\lambda_1 = r, \lambda_2 ,..., \lambda_h$ of absolute value r, then these numbers are distinct and satisfy $\lambda_j^h = r^h$ $(1 \leqslant j \leqslant h)$.*

See, for example, [3, pp. 65–75].

With the notation of Lemma 1, we call h the period of T and the eigenvalues $\lambda_1, \ldots, \lambda_h$ the dominant eigenvalues of T. In general, if T is a matrix of class $\mathscr{T}$ with the normal form (1), then the period of T is the least common multiple of the periods of $T_1, \ldots, T_\kappa$ and the dominant eigenvalues of T are those of absolute value $r(T)$.

LEMMA 2 [3, pp. 92–94]. *Let T be a matrix of class $\mathscr{T}$ with spectral radius r and period h. Then r is an eigenvalue of T and T has a positive eigenvector belonging to r. If λ is a dominant eigenvalue of T with multiplicity m, then $\lambda^h = r^h$ and T has m linearly independent eigenvectors belonging to λ.*

The restriction to matrices T in the class $\mathscr{T}$ ensures that the iterates of T grow uniformly in the sense of the following lemma.

LEMMA 3. *Let T be a matrix of class $\mathscr{T}$ with spectral radius r and period h. There is a decomposition $U \oplus V$ of the underlying space as a direct sum of invariant subspaces of T such that if $x = u + v$ with u in U and v in V, then*

$$T^{hk}x = r^{hk}u + O(r_0^{hk}) \qquad (k \to \infty),$$

where $0 \leqslant r_0 < r$ and both r_0 and the implied constant are independent of k.

Proof. Let $\lambda_1, \ldots, \lambda_m$ be the dominant eigenvalues of T. By elementary linear algebra and the facts established in Lemma 2, we can write

$$T = \sum_{j=1}^{m} \lambda_j E_j + F,$$

where E_j is the projection onto the eigenspace belonging to λ_j and r_0 (say) $= r(F) < r$. Moreover, $E_i E_j = \delta_{ij} E_j$ and $E_j F = F E_j = 0$, so

$$T^{hk} = \sum_{j=1}^{m} \lambda_j^{hk} E_j + O(r_0^{hk}) \qquad (k \to \infty).$$

The assertions of the lemma now follow on taking

$$U = \bigoplus_{j=1}^{m} \operatorname{im} E_j, \qquad V = \bigcap_{j=1}^{m} \ker E_j$$

and recalling from Lemma 2 that $\lambda_1{}^h = \cdots = \lambda_m{}^h = r^h$.

In the notation of Lemma 3, we will refer to the subspace U as the dominant eigenspace of T and to V as the residual subspace of T. We call the component u of the vector x in the lemma the projection of x on the dominant eigenspace of T.

2. *The Multiplicative Transformation*

Let $T = (t_{ij})$ be an $n \times n$ integer matrix. We define the multiplicative transformation $T: \mathbf{C}^n \to \mathbf{C}^n$ as follows: If $z = (z_1, ..., z_n)$ is a point of $\mathbf{C}^n$, then $w = Tz$ is the point with coordinates

$$w_i = \prod_{j=1}^{n} z_j^{t_{ij}} \qquad (1 \leqslant i \leqslant n).$$

Note that if T_1 and T_2 are two $n \times n$ integer matrices then, as the notation implies, $(T_1T_2)z = T_1(T_2z)$ for all z in $\mathbf{C}^n$.

We must now translate the results of Section 1 into the new notation. We denote by $\mathbf{C}^{*n}$ the set of points $z = (z_1, ..., z_n)$ in $\mathbf{C}^n$ with $z_1z_2 \cdots z_n \neq 0$. For z in $\mathbf{C}^{*n}$, we define $L(z)$ to be the real vector

$$L(z) = (-\log | z_1 |, ..., -\log | z_n |).$$

It is convenient also to adopt a vector notation for monomials: if $\mu = (\mu_1, ..., \mu_n)$ is a rational vector, then we write

$$z^\mu = z_1^{\mu_1} \cdots z_n^{\mu_n} \qquad (z \text{ in } \mathbf{C}^n).$$

Note that

$$(Tz)^\mu = z^{\mu T} \qquad (z \text{ in } \mathbf{C}^n),$$

where the exponent μT on the right is the usual product of the row vector μ and the matrix T. In the same spirit, if $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$, we write

$$| x | = \sum_{j=1}^{n} | x_j |$$

and

$$\langle x, y \rangle = \sum_{j=1}^{n} x_j y_j .$$

We can now restate Lemma 3.

LEMMA 4. *Let T be a matrix of class $\mathscr{T}$ with spectral radius r and period h. Let z be in $\mathbf{C}^{*n}$ and let u be the projection of $L(z)$ on the dominant eigenspace of T. Then*

$$L(T^{hk}z) = r^{hk}u + O(r_0^{hk}) \qquad (k \to \infty),$$

where $0 \leqslant r_0 < r$ and both r_0 and the implied constant are independent of k. Further, for any integral vector μ,

$$\log |(T^{hk}z)^\mu | = -r^{hk}\langle \mu, u \rangle + O(r_0^{hk}) \qquad (k \to \infty).$$

3. *Admissible Points*

In this section, we describe the admissible points, that is the points of $\mathbf{C}^n$ at which our transcendence theorems can be applied. Throughout this section, T denotes a matrix of class $\mathscr{T}$.

DEFINITION 2. We define $\mathscr{U}(T)$ to be the set of all points $z = (z_1, ..., z_n)$ in $\mathbf{C}^{*n}$ such that the projection of the vector $L(z) = (-\log | z_1 |, ..., -\log | z_n |)$ on the dominant eigenspace of T is positive. Thus $\mathscr{U}(T)$ is an open neighborhood of the origin in $\mathbf{C}^{*n}$.

DEFINITION 3. A point $\alpha = (\alpha_1, ..., \alpha_n)$ of $\mathbf{C}^n$ has Property (A) if, for every function $f(z)$ of n complex variables which is regular in some neighborhood of the origin and is not identically zero, there are infinitely many natural numbers k such that $f(T^k\alpha) \neq 0$.

Let $F(z)$ be a function of n complex variables which is regular in some neighborhood of the origin and satisfies the functional equation

$$F(Tz) = \sum_{j=0}^{s} a_j(z)\, F(z)^j \Big/ \sum_{j=0}^{s} b_j(z)\, F(z)^j, \tag{1}$$

where the coefficients $a_j(z)$ and $b_j(z)$ are polynomials in z. We denote by $\Delta(z)$ the resultant of the two forms $\sum a_j(z)\, u^j v^{s-j}$ and $\sum b_j(z)\, u^j v^{s-j}$, so that $\Delta(z)$ is also a polynomial in z.

DEFINITION 4. A point $\alpha = (\alpha_1, ..., \alpha_n)$ of $\mathbf{C}^{*n}$ is admissible (more explicitly, admissible with respect to the matrix T and the functional equation (1)) if it has the following three properties:

(i) α is in $\mathscr{U}(T)$;

(ii) $\Delta(T^k\alpha) \neq 0$ $(k = 0, 1, 2, ...)$; and

(iii) α has Property (A).

The first two conditions in Definition 4 are quite natural. For, after Lemma 4, condition (i) ensures that $T^k\alpha \to 0$ as $k \to \infty$, so that $T^k\alpha$ lies in the domain of regularity of $F(z)$ for all sufficiently large k. Also, from (1), we obtain

$$F(T^k z) = \sum_{j=0}^{s^k} a_j^{(k)}(z)\, F(z)^j \Big/ \sum_{j=0}^{s^k} b_j^{(k)}(z)\, F(z)^j \tag{2}$$

$(k = 1, 2, ...)$, where $a_j^{(k)}(z)$ and $b_j^{(k)}(z)$ are polynomials in z. The resultant of the forms $\sum a_j^{(k)}(z)\, u^j v^{s-j}$ and $\sum b_j^{(k)}(z)\, u^j v^{s-j}$ is easily seen to be $\Delta(z)\, \Delta(Tz) \cdots \Delta(T^{k-1}z)$. By condition (ii), all these resultants are nonzero at α, so that the rational functions on the right sides of (1) and (2) are well defined at α

whenever $F(\alpha)$ is defined. Condition (iii) is on quite a different footing, but we shall see in Part III that there are relatively unobjectionable criteria which guarantee that it holds.

II. The Transcendence Theorem

4. *Statement of the Transcendence Theorem*

This part will be taken up with the proof of the following theorem.

THEOREM 1. *Let T be a matrix of class $\mathscr{T}$ with spectral radius $r > 1$. Let $F(z) = \sum A_\mu z^\mu$ be a power series whose coefficients lie in some fixed algebraic number field and suppose that $F(z)$ converges in some neighborhood of the origin and satisfies the functional equation*

$$F(Tz) = \sum_{j=0}^{s} a_j(z)\, F(z)^j \Big/ \sum_{j=0}^{s} b_j(z)\, F(z)^j, \tag{1}$$

where the coefficients $a_j(z)$ and $b_j(z)$ are polynomials with algebraic coefficients and $s < r$. Finally, let α be an admissible algebraic point. If $F(z)$ is not an algebraic function and $F(\alpha)$ is defined, then $F(\alpha)$ is transcendental.

(For the notions of a matrix of class $\mathscr{T}$ and an admissible point, see Definition 1 of Section 1 and Definition 4 of Section 3, respectively.)

In a few simple cases, we have been able to show that the only algebraic solutions of the functional equation (1), if any, are of a very special type. For example, we show in [5, Theorem 2], that if T is a nonsingular matrix of class $\mathscr{T}$ having no roots of unity as eigenvalues and if $a(z)$ and $b(z)$ are rational functions, then any algebraic function $F(z)$ satisfying the functional equation

$$F(Tz) = a(z)\, F(z) + b(z) \tag{2}$$

must be rational. In particular, the examples given in the Introduction satisfy functional equations of the form (2) and are obviously not rational functions, so without any appeal to analytic continuation, we see that they are transcendental functions. We shall not pursue this point further here, but refer to [5] where more remarks on Eq. (2) may be found.

5. *The Auxiliary Function*

Let the matrix T, the power series $F(z)$, and the point α satisfy all the requirements of Theorem 1 and suppose, in addition, that $F(\alpha)$ is algebraic. Let K be an algebraic number field of finite degree, d say, over $\mathbf{Q}$ which contains all the coefficients of the power series $F(z)$, the coefficients of the polynomials $a_j(z)$ and $b_j(z)$ appearing in the functional equation, the coordinates of α and the number $F(\alpha)$.

For each β in K, we can find a nonzero rational integer den β, a denominator for β, such that (den $\beta)\beta$ is an algebraic integer. It is convenient to write

$$\| \beta \| = \max_{\sigma} \{| \sigma\beta |, | \operatorname{den} \beta |\},$$

where σ runs through the d distinct embeddings of K into $\mathbf{C}$. The eventual contradiction which gives us our result depends on the following inequality for $\| \beta \|$.

LEMMA 5. *Let K be an algebraic number field of degree d over $\mathbf{Q}$. If β is a nonzero algebraic number in K, then*

$$\log | \beta | \geqslant -2d \log \| \beta \|.$$

See, for example, [4, p. 3].

In the following work, $c_1, c_2, \ldots$ denote positive constants depending only on the quantities introduced above and, in particular, not depending on the parameters k and ρ which will appear shortly. If the context demands that, say, an exponent containing c_1 be a rational integer, then c_1 is assumed to be selected appropriately.

We begin the proof of Theorem 1 by constructing the auxiliary function.

LEMMA 6. *Let $F(z) = \sum A_\mu z^\mu$ be a power series whose coefficients lie in K and suppose that $F(z)$ converges in some neighborhood of the origin and does not represent an algebraic function. Then, for each $\rho \geqslant c_1$, there are $\rho + 1$ polynomials $p_0(z), \ldots, p_\rho(z)$ with degrees at most ρ in each variable and whose coefficients are algebraic integers in K, such that the function*

$$E_\rho(z) = \sum_{j=0}^{\rho} p_j(z) F(z)^j = \sum_{\mu} B_\mu z^\mu$$

is not identically zero, but all the coefficients B_μ with

$$| \mu | = \mu_1 + \cdots + \mu_n \leqslant \tfrac{1}{2}\rho^{1+1/n} \tag{1}$$

vanish.

Proof. The $\rho + 1$ polynomials $p_j(z)$ together possess $(\rho + 1)^{n+1}$ coefficients. On the other hand, the number of coefficients B_μ satisfying (1) is at most

$$(\tfrac{1}{2}\rho^{1+1/n} + 1)^n \leqslant (\rho + 1)^{n+1} - 1,$$

providing $\rho \geqslant c_1$. If we now require that all these coefficients B_μ vanish, then we have at most $(\rho + 1)^{n+1} - 1$ linear equations in $(\rho + 1)^{n+1}$ unknowns. This system has a nontrivial solution in K consisting of algebraic integers. Since $F(z)$ is not an algebraic function, it follows that the function $E_\rho(z)$ obtained in this way is not identically zero.

The proof of Theorem 1 will be completed in the next two sections as follows. We show first in Lemma 7 that

$$\log |E_\rho(T^k\alpha)| \leqslant -c_2 r^k \rho^{1+1/n},$$

providing $\rho \geqslant c_1$ and $k \geqslant c_3$. Then we show in Lemma 10 that

$$\log \| E_\rho(T^k\alpha)\| \leqslant c_4 r^k \rho,$$

providing $\rho \geqslant c_1$ and k is sufficiently large relative to ρ. Now, fix the parameter ρ by

$$\rho = c_5 \text{ (say)} > \max\{c_1, (2dc_4/c_2)^n\}.$$

The two estimates above combine to give

$$\log |E_\rho(T^k\alpha)| < -2d \log \| E_\rho(T^k\alpha)\|, \tag{2}$$

providing $k \geqslant c_6$. On the other hand, $E_\rho(T^k\alpha)$ is an algebraic number in K and it is nonzero for infinitely many k since α satisfies Property (A). For all such $k \geqslant c_6$, the inequality (2) contradicts Lemma 5. Thus $F(\alpha)$ cannot be an algebraic number and Theorem 1 is proved.

6. *An Upper Bound for* $|E_\rho(T^k\alpha)|$

LEMMA 7. *Let T be a matrix of class $\mathcal{T}$ with spectral radius $r > 1$. Let α be a point of $\mathcal{U}(T)$ and let $E_\rho(z)$ be the function constructed in Lemma 6. Then*

$$\log |E_\rho(T^k\alpha)| \leqslant -c_2 r^k \rho^{1+1/n}$$

providing $\rho \geqslant c_1$ and $k \geqslant c_3$.

Proof. Denote the period of T by h. Since α is in $\mathcal{U}(T)$, it follows from Lemma 4 that $T^{hk} \to 0$ as $k \to \infty$, so the series

$$E_\rho(T^{hk}\alpha) = \sum_\mu B_\mu (T^{hk}\alpha)^\mu$$

is convergent for all sufficiently large k. By Lemma 4 again,

$$\log |(T^{hk}\alpha)^\mu| \leqslant -c_7 r^{hk} |\mu|.$$

It follows, in the first place, that $\log |B_\mu| \leqslant c_8 |\mu|$ whenever $B_\mu \neq 0$, and so by Lemma 6,

$$\log |B_\mu (T^{hk}\alpha)^\mu| \leqslant -c_9 r^{hk} |\mu| \leqslant -\tfrac{1}{2} c_9 r^{hk} \rho^{1+1/n}$$

whenever $\rho \geqslant c_1$, $k \geqslant c_{10}$, and $B_\mu \neq 0$. Hence

$$\log | E_\rho(T^{hk}\alpha)| \leqslant -c_{11} r^{hk} \rho^{1+1/n}$$

providing $\rho \geqslant c_1$ and $k \geqslant c_{10}$. An analogous result holds with α replaced by $T^l\alpha$ $(1 \leqslant l \leqslant h-1)$. Combining the estimates obtained in this way gives the inequality of the lemma.

7. *An Upper Bound for* $\| E_\rho(T^k\alpha)\|$

For a polynomial $p(z) = \sum p_\mu z^\mu$ with coefficients in K, we define $\| p \| = \max \| p_\mu \|$. Further, we say the polynomial $q(z) = \sum q_\mu z^\mu$ dominates $p(z)$, written $p(z) \prec q(z)$, if the coefficients of $q(z)$ are rational integers and $\| p_\mu \| \leqslant q_\mu$ for each μ.

LEMMA 8. *Let T be a matrix of class $\mathscr{T}$ with spectral radius $r > 1$. Let $p(z) = \sum p_\mu z^\mu$ be a polynomial with coefficients in K and of degree at most ρ in each variable. Let α be a point with coordinates in K. Then*

$$\log \| p(T^k\alpha)\| \leqslant \log \| p \| + c_{12} r^k \rho.$$

Proof. Set

$$P(z) = \prod_{j=1}^{n} (1 + z_j). \tag{1}$$

The degree of the polynomial $P(T^k z)$ is at most $c_{13} r^k$, so

$$P(T^k z) \prec P(z)^{c_{13} r^k}. \tag{2}$$

Now

$$p(T^k z) \prec \| p \| \, P(T^k z)^\rho \prec \| p \| \, P(z)^{c_{13} r^k \rho},$$

which implies the inequality of the lemma.

LEMMA 9. *With the notation and hypotheses of Theorem 1, we have*

$$\log \| F(T^k\alpha)\| \leqslant c_{14} r^k$$

providing $k \geqslant c_{15}$.

Proof. We can write $F(z) = f(z)/g(z)$ where $f(z)$ and $g(z)$ are defined in some neighborhood of the origin and satisfy

$$\begin{aligned} f(Tz) &= \sum_{j=0}^{s} a_j(z) f(z)^j g(z)^{s-j}, \\ g(Tz) &= \sum_{j=0}^{s} b_j(z) f(z)^j g(z)^{s-j}, \end{aligned} \tag{3}$$

and both $f(\alpha)$ and $g(\alpha)$ are in K. As in Lemma 7, the point $T^k\alpha$ lies in the region of definition of f and g for all sufficiently large k. Denote the resultant of Eqs. (3), regarded as forms in $f(z)$ and $g(z)$, by $\Delta(z)$. From (3), we obtain inductively a similar pair of equations for $f(T^kz)$ and $g(T^kz)$ as forms in $f(z)$ and $g(z)$ whose resultant is $\Delta(z)\,\Delta(Tz)\cdots\Delta(T^{k-1}z)$ and so it follows, since α is an admissible point, that $f(T^k\alpha)$ and $g(T^k\alpha)$ are not both 0 for any k.

Define $P(z)$ again by (1). From (3),

$$f(Tz), g(Tz) \prec \tfrac{1}{2}c_{16}P(z)^{c_{17}}\{f(z)+g(z)\}^s,$$

where we regard $f(Tz)$ and $g(Tz)$ as polynomials in $z_1,\ldots,z_n$, $f(z)$, and $g(z)$. By induction on k and inequality (2),

$$f(T^kz), g(T^kz) \prec \tfrac{1}{2}c_{16}^{1+s+\cdots+s^{k-1}}P(z)^{c_{18}(r^{k-1}+sr^{k-2}+\cdots+s^{k-1})}\{f(z)+g(z)\}^{s^k}.$$

But $s < r$ and $f(\alpha)$ and $g(\alpha)$ are in K, so we have

$$\begin{aligned}\log\|F(T^k\alpha)\| &= \log\|f(T^k\alpha)/g(T^k\alpha)\|\\ &\leqslant \log\|f(T^k\alpha)\| + \log\|g(T^k\alpha)\|\\ &\leqslant c_{14}r^k\end{aligned}$$

providing $k \geqslant c_{15}$.

LEMMA 10. *Assume the notation and hypotheses of Theorem* 1 *and let* $E_\rho(z)$ *be the function constructed in Lemma* 6. *Then*

$$\log\|E_\rho(T^k\alpha)\| \leqslant c_4r^k\rho,$$

providing $\rho \geqslant c_1$ *and* k *is sufficiently large relative to* ρ.

Proof. Let $p_j(z)$ $(0 \leqslant j \leqslant \rho)$ be the polynomials constructed in Lemma 6 and set $p = \max\|p_j\|$. By Lemma 8,

$$\log\|p_j(T^k\alpha)\| \leqslant \log p + c_{12}r^k\rho \leqslant c_{19}r^k\rho,$$

providing k is sufficiently large relative to ρ. By Lemma 9, for $\rho \geqslant c_1$,

$$\log\|E_\rho(T^k\alpha)\| \leqslant c_{20}r^k\rho + \log\|F(T^k\alpha)^\rho\| \leqslant c_{21}r^k\rho,$$

again providing k is sufficiently large relative to ρ.

As explained in Section 5, this completes the proof of Theorem 1.

III. CRITERIA FOR THE NONVANISHING OF $f(T^k\alpha)$

8. *Statement of the Theorems*

Let T be a matrix of class $\mathscr{T}$. As before, we say a point α in $\mathbf{C}^n$ has Property (A) if, for every function $f(z)$ of n complex variables which is regular in some neighborhood of the origin and not identically zero, there are infinitely many natural numbers k such that $f(T^k\alpha) \neq 0$. For the application to Theorem 1, it is important to have some simple and fairly general criteria which imply that a point α satisfies Property (A). In this direction, we shall prove the following three theorems.

We say that the numbers $\alpha_1, \dots, \alpha_n$ are multiplicatively independent if the only relation

$$\alpha_1^{\mu_1} \cdots \alpha_n^{\mu_n} = 1$$

with integral exponents $\mu_1, \dots, \mu_n$ is the trivial relation given by $\mu_1 = \cdots = \mu_n = 0$.

THEOREM 2. *Let T be a matrix of class $\mathcal{T}$ with spectral radius greater than* 1 *and let $\alpha = (\alpha_1, \dots, \alpha_n)$ be an algebraic point of $\mathcal{U}(T)$. If $|\alpha_1|, \dots, |\alpha_n|$ are multiplicatively independent, then α has Property* (A).

THEOREM 3. *Let T be a matrix of class $\mathcal{T}$ with spectral radius greater than* 1 *and period* 1 *and suppose the characteristic polynomial of T is irreducible over* **Q**. *Then every point in $\mathcal{U}(T)$ has Property* (A).

THEOREM 4. *Let T be a lower triangular matrix of class $\mathcal{T}$ whose diagonal elements are all greater than* 1 *and let $\alpha = (\alpha_1, \dots, \alpha_n)$ be an algebraic point of $\mathcal{U}(T)$. If $\alpha_1, \dots, \alpha_n$ are multiplicatively independent, then α has Property* (A).

(For the notions of a matrix of class $\mathcal{T}$ and a point of $\mathcal{U}(T)$, see Definition 1 of Section 1 and Definition 2 of Section 3, respectively.)

Theorem 3 is given by Mahler [6]; we include it here for completeness. Theorem 4 extends a result of Mahler [7]. The proofs of Theorems 2 and 3 follow in Section 9 and the proof of Theorem 4, which is surprisingly more difficult, in Sections 10 and 11.

9. *Proofs of Theorems* 2 *and* 3

The proofs of Theorems 2 and 3 depend on the following lemma.

LEMMA 11. *Let T be a matrix of class $\mathcal{T}$ with spectral radius $r > 1$ and period h. Let $f(z) = \sum a_\mu z^\mu$ be a power series convergent in some neighborhood of the origin and not identically zero. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a point of $\mathcal{U}(T)$ and suppose that the coordinates of the projection, u say, of the vector*

$$L(\alpha) = (-\log|\alpha_1|, \dots, -\log|\alpha_n|)$$

on the dominant eigenspace of T are linearly independent over **Q**. *Then*

$$\log|f(T^{hk}\alpha)| \sim -cr^{hk} \qquad (k \to \infty)$$

with $c > 0$. *In fact,*

$$c = \min\{\langle \mu, u\rangle : a_\mu \neq 0\}. \tag{1}$$

Proof. Since α is in $\mathscr{U}(T)$, it follows as in the proof of Lemma 7, that the series $f(T^{hk}\alpha)$ is convergent for all sufficiently large k. Also from the defining property of $\mathscr{U}(T)$, we have $u > 0$, so the minimum in (1) is attained and is positive. Pick a nonzero term $a_m z^m$ of the series with $c = \langle m, u\rangle$. By Lemma 4,

$$\log | a_m(T^{hk}\alpha)^m | = -cr^{hk} + O(r_0^{hk}) \qquad (k \to \infty) \tag{2}$$

where $0 \leqslant r_0 < r$. If $a_\mu z^\mu$ is any other nonzero term of the series, then in the same way

$$\log | a_\mu(T^{hk}\alpha)^\mu / a_m(T^{hk}\alpha)^m | = -r^{hk}\langle \mu - m, u\rangle + O(r_0^{hk}) \qquad (k \to \infty),$$

and by the hypothesis on u and the choice of m, $\langle \mu - m, u\rangle > 0$. So for all sufficiently large k, the term $a_m(T^{hk}\alpha)^m$ is the dominant term of $f(T^{hk}\alpha)$ and the desired estimate follows from (2).

Proof of Theorem 2. After Lemma 11, it suffices to prove that the coordinates of the projection of $L(\alpha)$ on the dominant eigenspace of T are linearly independent over $\mathbf{Q}$. If, on the contrary, they are linearly dependent, then the coordinates $-\log | \alpha_1 |,..., -\log | \alpha_n |$ of $L(\alpha)$ itself are linearly dependent over the field of algebraic numbers and so, by a theorem of Baker [1], it follows that $1, \log | \alpha_1 |,..., \log | \alpha_n |$ are linearly dependent over $\mathbf{Q}$, contradicting the hypothesis of Theorem 2.

Proof of Theorem 3. Let T be a matrix of the type described in Theorem 3. Denote the spectral radius of T by r and let x be a positive eigenvector of T belonging to the eigenvalue r; the existence of such an x is given by Lemma 2. The coordinates $x_1 ,..., x_n$ of x satisfy the n homogeneous linear equations

$$(T - rI)x = 0. \tag{3}$$

If the coordinates of x are linearly dependent over $\mathbf{Q}$, we can replace one of the equations in system (3) by an equation

$$a_1 x_1 + \cdots + a_n x_n = 0$$

with rational coefficients such that the resulting system has a nontrivial solution. However, on equating the determinant of this system to zero, we have a contradiction to the hypothesis that r has degree n over $\mathbf{Q}$. Hence the coordinates of x are linearly independent over $\mathbf{Q}$. Now, T is irreducible in the sense of Section 1, since its characteristic polynomial is irreducible, and T has period 1 by hypothesis, so by Lemma 1, the dominant eigenspace of T is the one dimensional space spanned by x. It therefore follows from the first part of the proof, that any point α of $\mathscr{U}(T)$ satisfies the condition of Lemma 11 and this establishes the theorem.

10. *Lemmas about Polynomials*

The main step in the proof of Theorem 4 is provided by Lemmas 14 and 15 below. Before that, we need two preliminary remarks.

LEMMA 12. *Let T be an integral $n \times n$ matrix with spectral radius $r > 1$. Let $\theta = (\theta_1, \ldots, \theta_n)$ be an algebraic point of $\mathbf{C}^{*n}$ and $\mu = (\mu_1, \ldots, \mu_n)$ be a rational vector. Given $\epsilon > 0$, there is a positive number $k_0(\epsilon)$ such that if $k \geqslant k_0(\epsilon)$, then either*

$$\theta^{\mu T^k} = 1 \qquad \text{or} \qquad | \theta^{\mu T^k} - 1 | > e^{-\epsilon r^k}.$$

Proof. Set $\mu T^k = (\mu_1^{(k)}, \ldots, \mu_n^{(k)})$. The $\mu_j^{(k)}$ are rational numbers with absolute values and denominators bounded by $c_1 r^k$, say, where c_1 is a positive number independent of k. Let $\epsilon > 0$ and put

$$\Lambda = \mu_1^{(k)} \log \theta_1 + \cdots + \mu_n^{(k)} \log \theta_n - \log 1.$$

By a theorem of Baker [2], there is a positive number c_2, independent of k, such that either

$$\Lambda = 0 \qquad \text{or} \qquad | \Lambda | > c_2 e^{-\epsilon r^k}.$$

From this statement, we easily obtain the assertion of the lemma.

LEMMA 13 [5, Theorem 3]. *Let T be a nonsingular integral $n \times n$ matrix whose spectrum contains no roots of unity. Suppose $\phi(z) = \phi(z_1, \ldots, z_n)$ is an algebraic function of $z_1, \ldots, z_n$ and satisfies the functional equation*

$$\phi(Tz) = z^e \phi(z)^s,$$

where $e = (e_1, \ldots, e_n)$ is an integral vector and s is a positive integer. Then

$$\phi(z) = \rho z^c,$$

where $\rho^{s-1} = 1$ and c is a rational vector such that $c(T - sI) = e$.

LEMMA 14. *Let T be a lower triangular matrix of class $\mathscr{T}$ whose diagonal elements are all greater than 1. Let $\theta = (\theta_1, \ldots, \theta_n)$ be an algebraic point of $\mathbf{C}^{*n}$ such that the projection of the vector $L(\theta) = (-\log | \theta_1 |, \ldots, -\log | \theta_n |)$ on the dominant eigenspace of T is 0. Suppose that for some $\epsilon > 0$, there is a polynomial $f(z)$, not identically zero, such that*

$$f(T^k \theta) = O(e^{-\epsilon r^k}) \qquad (k \to \infty). \tag{1}$$

Then the coordinates of θ are multiplicatively dependent.

Proof. Let $\mathscr{I}$ be the ideal of all polynomials $f(z)$ which satisfy (1) for some $\epsilon > 0$. Let m be the smallest integer such that $\mathscr{I}$ contains a polynomial depending only on the variables $z_1, \ldots, z_m$ and let $f(z)$ be a nonzero polynomial in $\mathscr{I}$ with this property and having minimal degree, μ say, in z_m. Now $f(Tz)$ is also a polynomial in $z_1, \ldots, z_m$, so we can write

$$p(z) f(Tz) = q(z) f(z) + r(z),$$

where $p(z)$ is a polynomial in $z_1, \ldots, z_{m-1}$ and $q(z)$ and $r(z)$ are polynomials in $z_1, \ldots, z_m$ and $r(z)$ has degree less than μ in z_m. But $r(z)$ is in $\mathscr{I}$, so by construction, $r(z) = 0$, that is,

$$p(z) f(Tz) = q(z) f(z). \tag{2}$$

Partition $z = (z', z_m, z_{m+1}, \ldots, z_n)$. We can write

$$f(z) = \phi_0(z') \prod_{j=1}^{\mu} \{z_m - \phi_j(z')\}, \tag{3}$$

where $\phi_0(z')$ is a polynomial and the $\phi_j(z')$ $(1 \leqslant j \leqslant \mu)$ are algebraic. Similarly,

$$f(Tz) = \psi_0(z') \prod_{j=1}^{\nu} \{z_m - \psi_j(z')\}, \tag{4}$$

where $\psi_0(z')$ is a polynomial and the $\psi_j(z')$ $(1 \leqslant j \leqslant \nu)$ are algebraic. On the other hand, from (3),

$$\begin{aligned} f(Tz) &= \phi_0(w') \prod_{j=1}^{\mu} \{w_m - \phi_j(w')\} \\ &= z'^{t_m} \phi_0(w') \prod_{j=1}^{\mu} \{z_m^{t_{mm}} - z'^{-t_m} \phi_j(w')\}, \end{aligned} \tag{5}$$

where $t_m = (t_{m1}, t_{m2}, \ldots, t_{m,m-1})$ and we have written $w = Tz = (w', w_m, w_{m+1}, \ldots, w_n)$. From (2), (3), and (4), each

$$\phi_i(z') \qquad (1 \leqslant i \leqslant \mu)$$

is one of the

$$\psi_j(z') \qquad (1 \leqslant j \leqslant \nu)$$

and from (4) and (5), each

$$\psi_j(z')^{t_{mm}} \qquad (1 \leqslant j \leqslant \nu)$$

is one of the

$$z'^{-t_m} \phi_l(w') \qquad (1 \leqslant l \leqslant \mu).$$

Hence each $\phi_i(z')$ $(1 \leqslant i \leqslant \mu)$ satisfies a functional equation of the shape

$$\phi(T^k z') = z'^e \phi(z')^s$$

where e is an integral vector and s is a positive integer. By Lemma 13, we therefore have

$$\phi_i(z') = \rho_i z'^{c_i} \qquad (1 \leqslant i \leqslant \mu),$$

where each ρ_i is a root of unity and c_i is a rational vector. Hence

$$f(z) = z_m{}^{\mu} \phi_0(z') \prod_{j=1}^{\mu} \{1 - \rho_j z'^{c_j} z_m^{-1}\}. \tag{6}$$

By construction, $\phi_0(z')$ is not in $\mathscr{I}$. By Lemma 4 and our hypothesis that the projection of $L(\theta)$ on the dominant eigenspace of T is 0, we also see that the polynomial $z_m{}^{\mu}$ is not in $\mathscr{I}$. But $f(z)$ is in $\mathscr{I}$, so one of the factors in the product (6), say

$$g(z) = 1 - \rho_i z'^{c_i} z_m^{-1},$$

satisfies the inequality

$$| g(T^k \theta)| < e^{-\epsilon r^k / 3\mu}$$

for infinitely many values of k. Finally, Lemma 12 shows that $g(T^k\theta) = 0$ for some k, so that the coordinates of θ are multiplicatively dependent.

LEMMA 15. *Let T be a lower triangular nonnegative integer matrix whose diagonal elements lie strictly between* 1 *and r. Let $\theta = (\theta_1, \ldots, \theta_n)$ be an algebraic point of $\mathbf{C}^{*n}$. Suppose that for some $\epsilon > 0$, there is a polynomial $f(z)$, not identically zero, such that*

$$f(T^k \theta) = O(e^{-\epsilon r^k}) \qquad (k \to \infty).$$

Then the coordinates of θ are multiplicatively dependent.

Proof. Just as in the proof of Lemma 14, we can show that $f(z)$ has the shape (6) and the desired conclusion follows with only minor changes in our previous reasoning.

11. *Proof of Theorem* 4

Let T be a lower triangular integral matrix of class $\mathscr{T}$ with spectral radius r, all of whose diagonal entries are greater than 1. After making a permutation of the indices if necessary, we may suppose that T is in the normal form of Section 1, say

$$T = \begin{bmatrix} r & & & & & \\ & \ddots & & & 0 & \\ & 0 & r & & & \\ t_{\kappa+1,1} & \cdots & t_{\kappa+1,\kappa+1} & & & \\ \vdots & & & \ddots & & \\ t_{n,1} & & \cdots & & & t_{n,n} \end{bmatrix}.$$

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an algebraic point of $\mathscr{U}(T)$ and suppose $\alpha_1, \ldots, \alpha_n$ are multiplicatively independent. Let $u = (u_1, \ldots, u_n)$ be the projection of $L(\alpha) = (-\log|\alpha_1|, \ldots, -\log|\alpha_n|)$ on the dominant eigenspace of T. In the present case, the dominant eigenspace of T is the eigenspace of T belonging to the eigenvalue r, and we see that

$$u_i = -\log|\alpha_i| \qquad (1 \leqslant i \leqslant \kappa), \tag{1}$$

$$u_i = (r - t_{ii})^{-1} \sum_{j=1}^{i-1} t_{ij} u_j \qquad (\kappa + 1 \leqslant i \leqslant n). \tag{2}$$

After a further permutation of the indices if necessary, we may suppose that $\log|\alpha_1|, \ldots, \log|\alpha_m|$ form a maximal $\mathbf{Q}$-linearly independent subset of $\log|\alpha_1|, \ldots, \log|\alpha_\kappa|$. From (1) and (2), we can therefore write

$$u_{m+i} = \sum_{j=1}^{m} b_{ij} u_j \qquad (1 \leqslant i \leqslant n - m), \tag{3}$$

where the b_{ij} are rational. We set

$$\alpha_{m+i} = \alpha_1^{b_{i1}} \cdots \alpha_m^{b_{im}} \theta_i \qquad (1 \leqslant i \leqslant n - m). \tag{4}$$

By hypothesis, $\theta_1, \ldots, \theta_{n-m}$ are multiplicatively independent. Moreover,

$$|\theta_i| = 1 \qquad (1 \leqslant i \leqslant \kappa - m). \tag{5}$$

Let $f(z) = \sum a_\mu z^\mu$ be a power series convergent in some neighborhood of the origin and not identically zero. Write

$$f(z) = \sum_R f_R(z), \tag{6}$$

where

$$f_R(z) = \sum_{\langle \mu, u \rangle = R} a_\mu z^\mu;$$

the notation is chosen so that none of the $f_R(z)$ vanishes identically. Since $u > 0$, each $f_R(z)$ is a polynomial and the index of summation in series (6) runs through a discrete set, say

$$0 \leqslant R_0 < R_1 < R_2 < \cdots .$$

Set $\epsilon = \frac{1}{4}(R_1 - R_0) > 0$. As in the first part of the proof of Lemma 11, we see that

$$\sum_{\nu=1}^{\infty} f_{R_\nu}(T^k\alpha) = O(\exp\{-(R_1 - \epsilon)\, r^k\}) \qquad (k \to \infty). \tag{7}$$

Let S be the matrix formed from T by deleting its first m rows and columns. From (3) and (4) and the linear independence of $u_1, \ldots, u_m$ over $\mathbf{Q}$, we readily find that

$$f_R(T^k\alpha) = e^{-Rr^k} g_R(S^k\theta), \tag{8}$$

where $\theta = (\theta_1, \ldots, \theta_{n-m})$ and $g_R(w) = g_R(w_1, \ldots, w_{n-m})$ is a polynomial which does not vanish identically. We now consider two cases.

First case. Suppose $m = \kappa$. Then we can apply Lemma 15 to the matrix S, the point θ, and the polynomial $g_R(w)$, whence, by (8), there are infinitely many integers k such that

$$| f_{R_0}(T^k\alpha)| \geqslant \exp\{-(R_0 + \epsilon)\, r^k\}. \tag{9}$$

For all sufficiently large k in this sequence, we have from (7) and (9) that

$$\begin{aligned} | f(T^k\alpha)| &\geqslant | f_{R_0}(T^k\alpha)| - \left| \sum_{\nu=1}^{\infty} f_R(T^k\alpha) \right| \\ &\geqslant \exp\{-(R_0 + 2\epsilon)\, r^k\}. \end{aligned} \tag{10}$$

Second case. Suppose $m < \kappa$. In this case, we can apply Lemma 14 to the matrix S, the point θ and the polynomial $g_R(w)$, because by (5) and the analogs of (1) and (2) for S, the projection of $L(\theta)$ on the dominant eigenspace of S is 0. Thus, again, Eqs. (9) and (10) hold for infinitely many integers k.

This completes the proof that α has Property (A).

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